Exercise 2

Show that

$$\text{(a)}\quad \operatorname*{Res}_{z=\pi i} \frac{z-\sinh z}{z^2 \sinh z} = \frac{i}{\pi};$$
$$\text{(b)}\quad \operatorname*{Res}_{z=\pi i} \frac{\exp(zt)}{\sinh z} + \operatorname*{Res}_{z=-\pi i} \frac{\exp(zt)}{\sinh z} = -2\cos \pi t.$$

Solution

Part (a)

The residue at $z = i\pi$ can be calculated by

$$\operatorname*{Res}_{z=i\pi} \frac{z-\sinh z}{z^2 \sinh z} = \frac{p_1(i\pi)}{q_1'(i\pi)},$$

where $p_1(z)$ is set to be the function in the numerator and $q_1(z)$ is set to be the function in the denominator.

$$\begin{aligned} &p_1(z) = z - \sinh z & & & &\Rightarrow \quad p_1(i\pi) = i\pi - i\sin\pi = i\pi \\ &q_1(z) = z^2 \sinh z & &\rightarrow \quad q_1'(z) = 2z\sinh z + z^2\cosh z & &\Rightarrow \quad q_1'(i\pi) = 2(i\pi)i\sin\pi + (i\pi)^2\cos\pi = \pi^2 \end{aligned}$$

Therefore,

$$\operatorname*{Res}_{z=i\pi} \frac{z-\sinh z}{z^2 \sinh z} = \frac{i}{\pi}.$$

Part (b)

The residues at $z = i\pi$ and $z = -i\pi$ can be calculated by

$$\begin{aligned} \operatorname*{Res}_{z=\pi i} \frac{\exp(zt)}{\sinh z} &= \frac{p_2(i\pi)}{q_2'(i\pi)} \\ \operatorname*{Res}_{z=-\pi i} \frac{\exp(zt)}{\sinh z} &= \frac{p_2(-i\pi)}{q_2'(-i\pi)}, \end{aligned}$$

where $p_2(z)$ is set to be the function in the numerator and $q_2(z)$ is set to be the function in the denominator.

$$p_2(z) = \exp(zt) \qquad \Rightarrow \quad \begin{cases} p_2(i\pi) = \exp(i\pi t) \\ p_2(-i\pi) = \exp(-i\pi t) \end{cases}$$

$$q_2(z) = \sinh z \quad \rightarrow \quad q_2'(z) = \cosh z \quad \Rightarrow \quad \begin{cases} q_2'(i\pi) = \cosh i\pi = \cos\pi = -1 \\ q_2'(-i\pi) = \cosh(-i\pi) = \cos(-\pi) = -1 \end{cases}$$

Therefore,

$$\begin{aligned} \operatorname*{Res}_{z=\pi i} \frac{\exp(zt)}{\sinh z} + \operatorname*{Res}_{z=-\pi i} \frac{\exp(zt)}{\sinh z} &= \frac{\exp(i\pi t)}{-1} + \frac{\exp(-i\pi t)}{-1} \\ &= -\cos\pi t - \cancel{i\sin\pi t} - (\cos\pi t - \cancel{i\sin\pi t}) \\ &= -2\cos\pi t. \end{aligned}$$